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Approximate method for solving an elliptic problem with discontinuous coefficients

Dang Quang A *

Institute of Computer Science, Lieugiai-Badinh, Hanoi, Viet Nam

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Abstract

The paper deals with an elliptic problem with coefficients discontinuous along a certain surface. An approximate method for solving this problem is constructed. It is based on extrapolation of solutions of problems which contain a small parameter in the boundary condition. For solving the latter problems an effective iterative method is proposed. An estimate of total error of the approximate solution of the problem is obtained.

Key words: Elliptic problem; Discontinuous coefficients; Domain decomposition; Asymptotic expansion; Iterative method

1. Introduction

For solving problems with discontinuous coefficients homogeneous difference schemes are usually used. But the shortcoming of these schemes is the decrease of their accuracy in the neighbourhood of discontinuous surfaces. Therefore the treatment of discontinuities is one of the very important problems which are of interest to researchers. In [8] for an elliptic problem with discontinuous coefficients the domain decomposition method was employed.

It should be noticed that at present this method has attracted great attention from specialists in the field of computational mathematics (see [4,6] and the references therein). The first international symposium on this method for partial differential equations was held in 1988 in the United States.

In [8] an error estimate of order $O(1/N^{\alpha})$ of the approximate solution was obtained, where N is the iteration number and α is the constant characterizing the smoothness of the given functions.

* Present address: Institute of Information Technology, Nghia do, Tu liem, Hanoi, Viet Nam.

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In this paper, using the method of extrapolation of solutions of problems containing a small parameter in the boundary condition, we construct the approximate solution of the problem considered in [8] with the convergence rate of geometric progression. This method was proposed in our earlier work [3] for solving the Dirichlet problem for biharmonic equations.

Now we state the problem under consideration.

Let Ω be a bounded domain of the space $\mathbb{R}^m$ with Lipschitz boundary S, Ω_+ a strictly inner subdomain of Ω with Lipschitz boundary Γ. Denote by n the outward normal to Γ. Let $\Omega_- = \Omega \backslash \overline{\Omega}_+$.

Consider the following boundary value problem:

$$Lu \equiv \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - a(x)u = f(x), \quad x \in \Omega \backslash \Gamma, \tag{1}$$

$$a(x) \geqslant \delta > 0,$$

$$[u]_\Gamma = 0, \tag{2}$$

$$\left[\frac{\partial u}{\partial \nu} \right]_\Gamma = 0, \quad u|_S = \phi. \tag{3}$$

Here $[\cdot]_\Gamma$ denotes the jump of a function when passing through Γ, $\partial u^\pm / \partial \nu_\pm$ the conormal derivatives of $u^\pm$,

$$u^\pm(x) = u(x), \quad x \in \Omega_\pm,$$

$$\frac{\partial u^\pm}{\partial \nu_\pm} = \sum_{i,j=1}^{m} a_{ij}^\pm(x) \frac{\partial u^\pm}{\partial x_j} \cos(n, x_i).$$

Suppose that the functions $a_{ij}(x)$, $i, j = 1, \ldots, m$, $a(x)$ are sufficiently smooth in $\Omega_\pm$, have a discontinuity on Γ and satisfy the ellipticity condition in $\Omega_\pm$.

In the future, for the sake of simplicity in writing, we shall use the notations

$$\| f^\pm \|_S = \| f^\pm \|_{H^s(\Omega_\pm)}, \quad \| g \|_{s,\Gamma} = \| g \|_{H^s(\Gamma)},$$

where $H^s(\Omega_\pm)$, $H^s(\Gamma)$ are Sobolev spaces (see [5]).

2. Construction of an approximate solution

According to the original problem (1)–(3) we consider the following problem:

$$Lu_\epsilon = f(x), \quad x \in \Omega \backslash \Gamma, \tag{4}$$

$$\epsilon \left. \frac{\partial u_\epsilon^+}{\partial \nu_+} \right|_\Gamma + [u_\epsilon]_\Gamma = 0, \tag{5}$$

$$\left[\frac{\partial u_\epsilon}{\partial \nu} \right]_\Gamma = 0, \quad u_\epsilon|_S = \phi. \tag{6}$$

Note that the way of introducing parameter ϵ here essentially differs from that in [8].

The original problem (1)–(3) is a particular case of the general problem, whose theory of existence, uniqueness and smoothness of solution was studied in [2,9]. Below we shall establish qualitative results for problem (4)–(6) and at the same time re-establish known ones for (1)–(3).

2.1. Reduction of the problems to operator equations

Consider the linear operator K defined as follows:

$$K : g \to [w]_\Gamma,$$

where g is a function defined on Γ, w the solution of the problem

$$Lw = 0, \quad x \in \Omega \setminus \Gamma, \tag{7}$$

$$\left.\frac{\partial w^+}{\partial \nu_+}\right|_\Gamma = g, \tag{8}$$

$$\left[\frac{\partial w}{\partial \nu}\right]_\Gamma = 0, \quad w \mid_S = 0. \tag{9}$$

In [8] it was proved that K is positive, completely continuous in $L_2(\Gamma)$ and has a trivial kernel. Additionally assuming that $a_{ij}(x) = a_{ji}(x)$, $i, j = 1,\dots,m$, it is easy to show that the operator K is symmetric in $L_2(\Gamma)$. Moreover, from the theory of elliptic equations [5] it follows that K is a completely continuous mapping from $H^t(\Gamma)$ into $H^{t+1}(\Gamma)$, $t \geqslant 0$. More precisely, K is an isomorphism of spaces $H^t(\Gamma)$ and $H^{t+1}(\Gamma)$ (see [8]). We now reduce problems (1)–(3) and (4)–(6) to equations with the operator K.

For that purpose put

$$\left.\frac{\partial u^+}{\partial \nu_+}\right|_\Gamma = g. \tag{10}$$

Then for the solution of problem (1)–(3) we have the representation

$$[u]_\Gamma = Kg + Af + B\phi, \tag{11}$$

where A, B are certain linear operators.

Indeed, we can set

$$u = u_g + u_f + u_\phi, \tag{12}$$

where u_g, u_f, u_ϕ are solutions of the problems

$$Lu_g = 0, \quad x \in \Omega \setminus \Gamma, \qquad \left.\frac{\partial u_g^+}{\partial \nu_+}\right|_\Gamma = g, \qquad \left[\frac{\partial u_g}{\partial \nu}\right]_\Gamma = 0, \quad u_g \mid_S = 0, \tag{13}$$

$$Lu_f = f, \quad x \in \Omega \setminus \Gamma, \qquad \left.\frac{\partial u_f^+}{\partial \nu_+}\right|_\Gamma = 0, \qquad \left[\frac{\partial u_f}{\partial \nu}\right]_\Gamma = 0, \quad u_f \mid_S = 0, \tag{14}$$

$$Lu_\phi = 0, \quad x \in \Omega \setminus \Gamma, \qquad \left.\frac{\partial u_\phi^+}{\partial \nu_+}\right|_\Gamma = 0, \qquad \left[\frac{\partial u_\phi}{\partial \nu}\right]_\Gamma = 0, \quad u_\phi \mid_S = \phi. \tag{15}$$

As was defined above, we have

$$[u_g]_\Gamma = Kg.$$

From the problems (14) and (15) we see that u_f, u_ϕ linearly depend on f and ϕ respectively. Hence,

$$[u_f]_\Gamma = Af, \qquad [u_\phi]_\Gamma = B\phi,$$

where A, B are certain linear operators.

Using the obvious equality $[u]_\Gamma = [u_g]_\Gamma + [u_f]_\Gamma + [u_\phi]_\Gamma$ and (2), we obtain the equation

$$Kg = F, \tag{16}$$

where

$$F = -Af - B\phi. \tag{17}$$

The smoothness of F depends on that of f and ϕ. Namely, using the theory of elliptic problems [5], it is easy to show that if $f^\pm \in H^{n-2}(\Omega_\pm)$, $\phi \in H^{n-1/2}(\Gamma)$, then $F \in H^{n-1/2}(\Gamma)$. Thus we reduced the original problem (1)–(3) to the operator equation (16) in the Hilbert space.

Similarly, putting

$$\left.\frac{\partial u_\epsilon^+}{\partial \nu_+}\right|_\Gamma = g_\epsilon, \tag{18}$$

we reduce problem (4)–(6) to the equation

$$(K + \epsilon I) g_\epsilon = F, \tag{19}$$

where I is the identity operator.

Theorem 1. *Let $f^\pm \in H^{n-2}(\Omega_\pm)$, $\phi \in H^{n-1/2}(\Gamma)$, $n \geqslant 2$. Then,*

(i) *equations* (16) *and* (19) *have unique solutions*

$$g \in H^{n-3/2}(\Gamma), \qquad g_\epsilon \in H^{n-1/2}(\Gamma);$$

(ii) *problems* (1)–(3) *and* (4)–(6) *have unique solutions u, u_ϵ such that $u^\pm \in H^n(\Omega_\pm)$, $u_\epsilon^\pm \in H^n(\Omega_\pm)$.*

Proof. Under the assumption of the theorem, as was noted above, we have $F \in H^{n-1/2}(\Gamma)$. Hence (16) has a unique solution $g \in H^{n-3/2}(\Gamma)$ because K is an isomorphism of spaces $H^{n-3/2}(\Gamma)$ and $H^{n-1/2}(\Gamma)$.

Now we consider (19) in $L_2(\Gamma)$. In this space we have $(K + \epsilon I)^* = K + \epsilon I > \epsilon I$. Therefore, the equation has a unique solution $g_\epsilon \in L_2(\Gamma)$. We shall show that $g_\epsilon \in H^{n-1/2}(\Gamma)$.

Indeed, from (19) it follows that

$$g_\epsilon = \frac{1}{\epsilon}(F - Kg_\epsilon).$$

Since $g_\epsilon \in L_2(\Gamma) = H^0(\Gamma)$ we have $Kg_\epsilon \in H^1(\Gamma)$. Hence $F - Kg_\epsilon \in H^{t_1}(\Gamma)$, where $t_1 = \min(n - \frac{1}{2}, 1)$. It implies $g_\epsilon \in H^{t_1}(\Gamma)$. Repeating the above argument k times, we arrive at the fact that

$g_\epsilon \in H^{t_k}(\Gamma)$, where $t_k = \min(n - \frac{1}{2}, k)$. Since K is an arbitrary natural number, we conclude that $g_\epsilon \in H^{n-1/2}(\Gamma)$. Thus, the first assertion of the theorem is proved.

The second assertion of the theorem follows from the fact that u and u_ϵ are solutions of the equivalent problems

$$Lu = f, \quad x \in \Omega \backslash \Gamma, \qquad \left.\frac{\partial u^+}{\partial \nu_+}\right|_\Gamma = g, \qquad \left[\frac{\partial u}{\partial \nu}\right]_\Gamma = 0, \quad u\,|_S = \phi,$$

$$Lu_\epsilon = f, \quad x \in \Omega \backslash \Gamma, \qquad \left.\frac{\partial u_\epsilon^+}{\partial \nu_+}\right|_\Gamma = g_\epsilon, \qquad \left[\frac{\partial u_\epsilon}{\partial \nu}\right]_\Gamma = 0, \quad u_\epsilon\,|_S = \phi,$$

where g, g_ϵ are the solutions of the equations (16)–(19), and from the theory of elliptic problems. □

2.2. Asymptotic expansion of the solution of problem (4)–(6)

In the sequel we denote by C, C^*, C_i, $i = 1, 2, \ldots,$ constants independent of ϵ.

Theorem 2. *Let $f^\pm \in H^{n-2}(\Omega_\pm)$, $\phi \in H^{n-1/2}(\Gamma)$, $n \geqslant 3$. Then for the solution of problem* (4)–(6) *there holds the asymptotic expansion*

$$u_\epsilon = \sum_{i=0}^{N} \epsilon^i v_i + \epsilon^{N+1} w_\epsilon, \quad x \in \Omega \backslash \Gamma, \ 0 \leqslant N \leqslant n-3, \tag{20}$$

where $v_0 = u$ is the solution of problem (1)–(3), *v_i are functions independent of ϵ, $v_i^\pm \in H^{n-i}(\Omega_\pm)$, $i = 1, \ldots, N$, $w_\epsilon^\pm \in H^{n-N}(\Omega_\pm)$ and*

$$\| w_\epsilon^\pm \|_{3/2} \leqslant C. \tag{21}$$

Proof. By Theorem 1 we have $v_0^\pm \in H^n(\Omega_\pm)$. After substituting (20) into (4)–(6) and balancing coefficients of powers of ϵ, we see that v_i and w_ϵ satisfy the following problems:

$$\begin{gathered} Lv_i = 0, \quad x \in \Omega \backslash \Gamma, \qquad [v_i]_\Gamma = -\left.\frac{\partial v_{i-1}^+}{\partial \nu_+}\right|_\Gamma, \\ \left[\frac{\partial v_i}{\partial \nu}\right]_\Gamma = 0, \quad v_i\,|_S = 0, \ i = 1, \ldots, N, \end{gathered} \tag{22}$$

$$\begin{gathered} Lw_\epsilon = 0, \quad x \in \Omega \backslash \Gamma, \qquad \epsilon \left.\frac{\partial w_\epsilon^+}{\partial \nu_+}\right|_\Gamma + [w_\epsilon]_\Gamma = -\left.\frac{\partial v_N^+}{\partial \nu_+}\right|_\Gamma, \\ \left[\frac{\partial w_e}{\partial \nu}\right]_\Gamma = 0, \quad w_\epsilon\,|_S = 0. \end{gathered} \tag{23}$$

Using the results of [2,9] or by reducing to boundary operator equations as was done above, we successively obtain that problem (22) has unique solutions $v_i^\pm \in H^{n-i}(\Omega_\pm)$, $i = 1, \ldots, N$.

Problem (23) may be reduced to the equation

$$(K+\epsilon I)h_\epsilon=\psi, \tag{24}$$

where

$$h_\epsilon=\left.\frac{\partial w_\epsilon^+}{\partial \nu_+}\right|_\Gamma, \qquad \psi=-\left.\frac{\partial v_N^+}{\partial \nu_+}\right|_\Gamma. \tag{25}$$

Since $v_N^+\in H^{n-N}(\Omega_+)$, we have $\psi\in H^{n-N-3/2}(\Gamma)$. Using a similar argument as in the proof of Theorem 1, we conclude that (24) has a unique solution $h_\epsilon\in H^{n-N-3/2}(\Gamma)$.

Since w_ϵ satisfies the problem

$$Lw_\epsilon=0,\quad x\in\Omega\setminus\Gamma,\qquad \left.\frac{\partial w_\epsilon^+}{\partial \nu_+}\right|_\Gamma=h_\epsilon,\qquad \left[\frac{\partial w_\epsilon}{\partial \nu}\right]_\Gamma=0,\quad w_\epsilon|_S=0,$$

we have (see [5])

$$\| w_\epsilon^\pm \|_t \leqslant C_1 \| h_\epsilon \|_{t-3/2,\Gamma}. \tag{26}$$

According to (24) we consider the equation

$$Kh_0=\psi.$$

Since K is an isomorphism of spaces $H^{t-3/2}(\Gamma)$ and $H^{t-1/2}(\Gamma)$, this equation has a unique solution $h_0\in H^{n-N-5/2}(\Gamma)$. However, we shall consider h_0 and h_ϵ as elements of $L_2(\Gamma)$. Then by Lemma 7 (in the Appendix) we have

$$\| h_\epsilon \|_{0,\Gamma}\leqslant \| h_0\|_{0,\Gamma},$$

whence, and from (26) with $t=\frac{3}{2}$, we obtain

$$\| w_\epsilon^\pm \|_{3/2}\leqslant C_1\| h_0\|_{0,\Gamma}.$$

Thus, estimate (21) is established with $C=C_1\| h_0\|_{0,\Gamma}$. The theorem is proved. □

2.3. Construction of the approximate solution by extrapolation

Put

$$U^{\mathrm{E}}=\sum_{i=1}^{N+1}\gamma_i u_{\epsilon/i}, \tag{27}$$

where $u_{\epsilon/i}$, $i=1,\dots,N+1$ are the solutions of problem (4)–(6) with parameter ϵ/i,

$$\gamma_i=\frac{(-1)^{N+1-i}i^{N+1}}{i!\,(N+1-1)!}. \tag{28}$$

We shall take U^{E} to be an approximate solution of the original problem (1)–(3).

Theorem 3. *Let the assumption of Theorem 2 be satisfied. Then for the approximate solution U^{E} we have the estimate*

$$\| U^{\mathrm{E}\pm}-u^\pm\|_{3/2}\leqslant C_2\epsilon^{N+1}. \tag{29}$$

Proof. For each $u_{\epsilon/i}$, $i=1,\dots,N+1$, we write the expansion (20) and take its sum with weights γ_i. By Lemma 8 (in the Appendix) we have

$$U^{\mathrm{E}}-u=\epsilon^{N+1}\sum_{i=1}^{N+1}\frac{\gamma_i}{i^{N+1}}w_{\epsilon/i}.$$

Using estimate (21), we get (29), where

$$C_2=C\sum_{i=1}^{N+1}\frac{|\gamma_i|}{i^{N+1}}.$$

The theorem is proved. □

3. Iterative method for solving problem (4)–(6)

In order to construct an iterative method for solving (4)–(6), we use the two-layer iterative scheme for the operator equation (19), to which the problem is reduced. The iterative process is defined as follows:

$$\frac{g_\epsilon^{(k+1)}-g_\epsilon^{(k)}}{\tau_\epsilon^{(k+1)}}+(\epsilon I+K)g_\epsilon^{(k)}=F,\quad k=0,\dots,M-1, \tag{30}$$

$$g_\epsilon^{(0)}\in L_2(\Gamma),$$

where $\{\tau_\epsilon^{(k+1)}\}$ is the Chebyshev collection of parameters [10], defined by the formulas

$$\tau_\epsilon^{(0)}=\frac{2}{\gamma_\epsilon^{(1)}+\gamma_\epsilon^{(2)}},\qquad \tau_\epsilon^{(k)}=\frac{\tau_\epsilon^{(0)}}{\varrho_\epsilon t_k+1},\quad t_k=\cos\frac{2k-1}{2M}\pi,$$

$$\varrho_\epsilon=\frac{1-\xi_\epsilon}{1+\xi_\epsilon},\qquad \xi_\epsilon=\frac{\gamma_\epsilon^{(1)}}{\gamma_\epsilon^{(2)}},\qquad \gamma_\epsilon^{(1)}=\epsilon,\qquad \gamma_\epsilon^{(2)}=\epsilon+\|K\|. \tag{31}$$

According to the general theory of two-layer iterative schemes [10], we have

$$\|g_\epsilon^{(k)}-g_\epsilon\|_{0,\Gamma}\leqslant q_{\epsilon,M}\|g_\epsilon^{(0)}-g_\epsilon\|_{0,\Gamma}, \tag{32}$$

where g_ϵ is the exact solution of (19),

$$q_{\epsilon,M}=\frac{2\left(\varrho_\epsilon^{(1)}\right)^M}{1+\left(\varrho_\epsilon^{(1)}\right)^{2M}},\quad \varrho_\epsilon^{(1)}=\frac{1-\sqrt{\xi_\epsilon}}{1+\sqrt{\xi_\epsilon}}.$$

In the case of the simple iteration $\tau_\epsilon^{(k)}=\tau_\epsilon^{(0)}$, $k=1,2,\dots,$ instead of (32), we get

$$\|g_\epsilon^{(k)}-g_\epsilon\|_{0,\Gamma}\leqslant(\varrho_\epsilon)^k\|g_\epsilon^{(0)}-g_\epsilon\|_{0,\Gamma}. \tag{33}$$

Remark 4. The above results on the convergence of the iterative scheme (30) are obtained only in the case of operator K being symmetric (i.e., when $a_{ij}(x)=a_{ji}(x)$). This is the assumption which ensures the existence of an orthonormal basis of $L_2(\Gamma)$, consisting of the eigenfunctions of K in [8, proof of Lemma 2]; consequently the main result of that paper is obtained, although this assumption was not mentioned there.

In the case of the operator K not being symmetric, it is possible to choose iterative parameters so that the iterative process (30) converges and there holds an estimate of the type (33) (see [10]). However, in this case we do not know whether or not Theorem 2 on asymptotic expansions remains valid.

The iterative process (30) can be realized by the following algorithm.

(1) Give a starting approximation $g_\epsilon^{(0)} \in L_2(\Gamma)$.

(2) Knowing $g_\epsilon^{(k)}$, $k=0,\dots,M-1$, solve the problem

$$Lu_\epsilon^{(k)}=f, \quad x\in\Omega\backslash\Gamma, \qquad \left.\frac{\partial u_\epsilon^{(k)+}}{\partial\nu_+}\right|_\Gamma = g_\epsilon^{(k)}, \qquad \left[\frac{\partial u_\epsilon^{(k)}}{\partial\nu}\right]_\Gamma = 0, \quad u_\epsilon^{(k)}|_S=0.$$

This problem is split into two Neumann problems in subdomains Ω_+ and Ω_-.

(3) Compute $[u_\epsilon^{(k)}]_\Gamma$ and $\delta_\epsilon^{(k)} = [u_\epsilon^{(k)}]_\Gamma + \epsilon g_\epsilon^{(k)}$.

(4) Compute the new approximation

$$g_\epsilon^{(k+1)} = g_\epsilon^{(k)} - \tau_\epsilon^{(k+1)}\delta_\epsilon^{(k)}.$$

Now we obtain the error estimate of the approximate solution $u_\epsilon^{(k)}$. Put $z_\epsilon^{(k)} = u_\epsilon^{(k)} - u_\epsilon$. We see that $z_\epsilon^{(k)}$ satisfies the boundary value problem

$$Lz_\epsilon^{(k)}=0, \quad x\in\Omega\backslash\Gamma, \qquad \left.\frac{\partial z_\epsilon^{(k)+}}{\partial\nu_+}\right|_\Gamma = g_\epsilon^{(k)} - g_\epsilon, \qquad \left[\frac{\partial z_\epsilon^{(k)}}{\partial\nu}\right]_\Gamma = 0, \quad z_\epsilon^{(k)}|_S=0.$$

Therefore, we have

$$\| z_\epsilon^{(k)} \|_{3/2} \leqslant C_3 \| g_\epsilon^{(k)} - g_\epsilon \|_{0,\Gamma}.$$

From this estimate and (33) we get

$$\| u_\epsilon^{(k)\pm} - u_\epsilon^\pm \|_{3/2} = \| z_\epsilon^{(k)\pm} \|_{3/2} \leqslant C_3(\varrho_\epsilon)^k \| g_\epsilon^{(0)} - g_\epsilon \|_{0,\Gamma}, \quad k=1,2,\dots\,. \tag{34}$$

If in (30) we take $g_\epsilon^{(0)} = g^{(0)} \in L_2(\Gamma)$, then we have

$$\| g_\epsilon^{(0)} - g_\epsilon \|_{0,\Gamma} = \| g^{(0)} - g_\epsilon \|_{0,\Gamma} \leqslant \| g^{(0)} \|_{0,\Gamma} + \| g_\epsilon \|_{0,\Gamma}. \tag{35}$$

By virtue of Lemma 7 (in the Appendix) we have

$$\| g_\epsilon \|_{0,\Gamma} \leqslant \| g \|_{0,\Gamma},$$

where g is the solution of the equation $Kg=F$. Hence from (35) it follows that

$$\| g_\epsilon^{(0)} - g_\epsilon \|_{0,\Gamma} \leqslant C_4, \tag{36}$$

where $C_4 = \| g^{(0)} \|_{0,\Gamma} + \| g \|_{0,\Gamma}$.

Finally from (34) and (36) we obtain

$$\| u_\epsilon^{(k)\pm} - u_\epsilon^\pm \|_{3/2} \leqslant C_5(\varrho_\epsilon)^k, \quad k=1,2,\dots, \tag{37}$$

where $C_5 = C_3C_4$ and ϱ_ϵ is defined by (31).

4. Conclusion

In order to obtain an approximate solution of the original problem (1)–(3) with the given accuracy ϵ^* we suggest the following algorithm.

(i) Choose ϵ such that

$$\epsilon^{N+1} = \epsilon^*, \tag{38}$$

where N is defined in Theorem 2.

(ii) For $i = 1, \ldots, N+1$ solve problem (4)–(6) with parameter ϵ/i by the algorithm described in Section 3. The process will be finished if after k_i iterations there holds

$$(\varrho_{\epsilon/i})^{k_i} \leqslant \epsilon^*. \tag{39}$$

As a real approximate solution of problem (1)–(3) we take

$$\mathscr{U}^{\mathrm{E}} = \sum_{i=1}^{N+1} \gamma_i u_{\epsilon/i}^{(k_i)}.$$

Then we have the error estimate

$$\| \mathscr{U}^{\mathrm{E}\pm} - u^{\pm} \|_{3/2} \leqslant C_6 \epsilon^*, \tag{40}$$

where C_6 is a constant independent of ϵ^*. Indeed, we have

$$\begin{aligned}
\| \mathscr{U}^{\mathrm{E}\pm} - u^{\pm} \|_{3/2} &= \| \sum_{i=1}^{N+1} \gamma_i u_{\epsilon/i}^{(k_i)\pm} - u^{\pm} \|_{3/2} \\
&= \| \sum_{i=1}^{N+1} \gamma_i u_{\epsilon/i}^{\pm} - u^{\pm} + \sum_{i=1}^{N+1} \gamma_i \left(u_{\epsilon/i}^{(k_i)\pm} - u_{\epsilon/i}^{\pm} \right) \|_{3/2} \\
&\leqslant \| U^{\mathrm{E}\pm} - u^{\pm} \|_{3/2} + \sum_{i=1}^{N+1} | \gamma_i | \, \| \left(u_{\epsilon/i}^{(k_i)\pm} - u_{\epsilon/i}^{\pm} \right) \|_{3/2} \\
&\leqslant C_2 \epsilon^{N+1} + C_5 \sum_{i=1}^{N+1} | \gamma_i | (\varrho_{\epsilon/i})^{k_i},
\end{aligned}$$

in view of (30) and (37).

Using (38) and (39), we get (40) with

$$C_6 = C_2 + C_5 \sum_{i=1}^{N+1} | \gamma_i |.$$

Obviously, C_6 does not depend on ϵ^*.

Remark 5. It is possible to set problem (1)–(3) for the case when the surface Γ divides the domain Ω into two parts Ω_+ and Ω_-, which have common boundary parts S_+ and S_- with Ω, $S = S_+ \cup S_-$.

Then the boundary condition on the interface S may be

(i) $\partial u / \partial \nu |_S = \psi$,

(ii) $u |_S = \phi$.

In case (ii) the assumption $a(x) \geqslant \delta > 0$ is not necessary.

The operator analogous to K was studied in [1] and is called the Poincaré–Steklov operator.

Remark 6. Our algorithm for solving elliptic problems with discontinuous coefficients is optimally carried out on parallel computers.

Appendix

Lemma 7. *Suppose the A is a linear, symmetric, positive and completely continuous operator in the Hilbert space H with the scalar product $(\cdot, \cdot)$. Let $f \in R(A)$, $\epsilon > 0$ and let u_ϵ, u_0 be the solutions of the equations*

$$\epsilon u_\epsilon + A u_\epsilon = f, \tag{41}$$

$$A u_0 = f. \tag{42}$$

Then the following estimate holds:

$$\| u_\epsilon \| \leqslant \| u_0 \|. \tag{43}$$

Proof. By virtue of the properties of the operator A there exists an orthonormal basis $e_1, e_2, \dots, e_n, \dots$ consisting of the eigenvectors of A.

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots > 0$ be the corresponding eigenvalues of A, i.e., $Ae_n = \lambda_n e_n$, $n = 1, 2, \dots$. Let

$$f = \sum_{n=1}^{\infty} f_n e_n, \qquad u_\epsilon = \sum_{n=1}^{\infty} a_n^{(\epsilon)} e_n, \qquad u_0 = \sum_{n=1}^{\infty} a_n^{(0)} e_n.$$

Then from equations (41) and (42) it follows that

$$a_n^{(\epsilon)} = \frac{f_n}{\epsilon + \lambda_n}, \quad a_n^{(0)} = \frac{f_n}{\lambda_n}, \qquad n = 1, 2, \dots .$$

Hence,

$$\| u_\epsilon \|^2 = \sum_{n=1}^{\epsilon} | a_n^{(\epsilon)} |^2 = \sum_{n=1}^{\infty} \frac{| f_n |^2}{(\epsilon + \lambda_n)^2} < \sum_{n=1}^{\infty} \frac{| f_n |^2}{\lambda_n^2} = \sum_{n=1}^{\infty} | a_n^{(0)} |^2 = \| u_0 \|^2,$$

because $\epsilon > 0$, $\lambda_n > 0$, $n = 1, 2, \dots$.

The lemma is proved. □

Lemma 8 (see [7, Appendix]). *If γ_i, $i = 1, \dots, N+1$, are defined by* (28), *then*

$$\sum_{i=1}^{N+1} \gamma_i = 1, \qquad \sum_{i=1}^{N+1} \gamma_i \frac{1}{i^l} = 0, \quad l = 1, \dots, N.$$

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